Maximum Entropy Magnetic Field Modelling

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Introduction

- High quality data available from ∅ersted, Champ, SAC-C and the forthcoming SWARM mission.
  - Potential SV recovery up to higher l than existing models.
  - Need better magnetic field maps at CMB for use with high resolution SV to produce core flow maps.
  - Crustal masking above $l = 13$.
  - Use Maximum Entropy technique following Jackson (2003) - circumvents crustal masking?
- Maximum Entropy
  - Common in fields such as astronomy.
  - Produces sharper images with stronger gradients and with less bias against smaller scales than commonly used quadratic regularisation.
  - Maximum Entropy produces more realistic spectra than quadratic regularisation with contribution from intermediate and small scales.
- Starting point is not data but the CHAOS model of Olsen et al (2006).
  - CHAOS model derived from 6.5 years of satellite data from CHAMP, ∅ersted and SAC-C.
  - The model parameterises external field in modelling process.
  - Thus this study's technique is purely the separation of crustal and core fields.
  - No need to prescribe error. Large solution space can be investigated.
  - Can incorporate crustal variance to weight the misfit to the CHAOS model.

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Maximum Entropy

- Maximum Entropy is a image processing technique first used in geomagnetism by Jackson (2003).
- Concept
  - Image is generated by throwing balls (intensity) into boxes (pixels).
  - The entropy of the image is defined by the natural logarithm of the number of ways the image can be produced.
  - Useful to introduce a default image which provides a priori information. The image with maximum entropy is now the default image.
- For a positive and negative distribution of B (the radial magnetic field) with data points i and using a flat default map (with points f_o) the entropy E can be defined (Hobson and Lasenby (1998))

$$E(B_i) = \sum_i \left\{ \psi_i - 2f_o - B_i \log\left[\frac{\psi + B_i}{2f_o}\right] \right\} \qquad \psi = (B_i^2 + 4f_o^2)^{1/2}$$

- Objective Function $\int_S (\underline{B} - \underline{B}_o)^2 d\Omega + \mu \int_{CMB} E \, d\Omega$
- As the entropy is non-linear in B the objective function (for coefficients X_n) takes it form from linearising the residuals and is minimised to give

$$0 = 2(l+1)(\Delta X_n + X_n - X_n^o) - \mu \frac{dE}{dX}$$

- Coefficients of dE/dX were formed using a spherical transform and the solution found iteratively until the model converges. Full Step did not converge – Fractional step was used. Depending on the value of μ different models were produced.

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Crustal Models

- Starting point is CHAOS model, purely separation of crustal and core fields.
- Utilise this advantage by weighting misfit by a crustal model.

What crustal model to choose?

- We consider 4 Choices
  - 2 Statistical following Jackson (1994).
  - 2 Predicted from magnetic and geological data (Fox Maule et al 2005 and Hemant and Maus 2005).
- From figure 1 the chosen model is Jackson (1994) k3 model.
  - It is a smoother model than the predicted models and lies between them.
  - Predicted models fit the data to a certain degree around interesting intermediate scales, thus they could be distorted.
  - k3 fits predictive models more closely than k4.
- Statistical models of Jackson (1994) are based on a correlation function of the relationship between magnetic fields and magnetisation of source region.

Lowes' power Spectrum

$$\langle B \rangle^2 = \sum_{l=1}^{\infty} R_l = \sum_{l=1}^{\infty} (l+1) \sum_{m=0}^{l} ((g_l^m)^2 + (h_l^m)^2)$$

K3 defined by

$$R_l = \kappa(\beta) \frac{l(l+1)}{(2l-1)} v^{l-1} K(l, \varepsilon)$$

$$K(l,\varepsilon) = (1-(1-\varepsilon)^{2l+1}) \qquad \varepsilon = d / a$$

Here κ is a constant as is v, l is harmonic degree, d is crustal thickness and a is the Earth's radius. κ =6.5 v=0.995 d=35km a=6371.2 for chosen k3 model.

Using crustal variance gives new equation to solve:

$$\Delta X_n = \frac{(2l+1)(X_n^o - X_n)}{R_l} + \lambda E_n$$

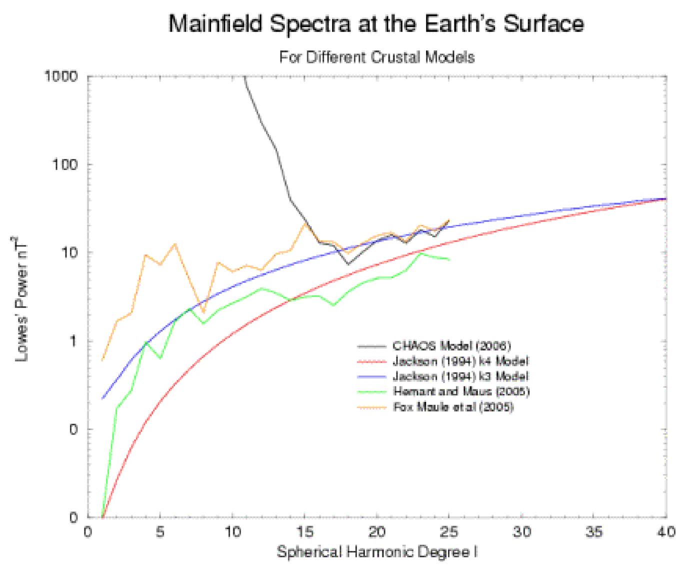

Figure 1. Crustal model spectra.

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Quadratic Regularisation

- Widely used in core field modelling (Langel and Estes 1982).
- As it is a quadratic measure can be thought of as a measure of magnetic energy.
- Solution is smooth and well behaved at the CMB.
- Heavily damps intermediate and small scales, affectively working as a taper.
- Cuts off crustal signal.

Objective Function

$$\int_S (\underline{B} - \underline{B}_o)^2 d\Omega + \mu \int_{CMB} B^2 d\Omega$$

Minimise by differentiating, setting to zero and rearranging

$$X_n = \frac{X_n^o}{1 + \mu (a/c)^{2l+4}}$$

Solve using least squares method of inversion

$$\underline{x} = \left(\underline{A}^T \underline{C}_e^{-1} \underline{A} + \mu \underline{I}\right)^{-1} \left(\underline{A}^T \underline{C}_e^{-1} \underline{d}\right)$$

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Model Solutions

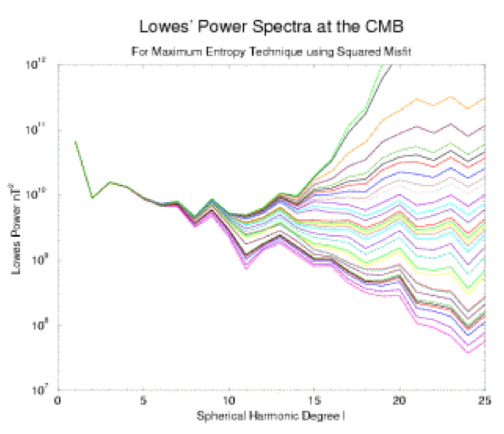


Figure 2 is using the Maximum Entropy technique with squared misfit.

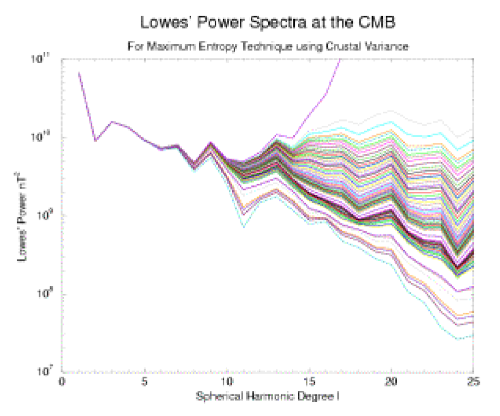


Figure 3 is the Maximum Entropy technique using the Jackson's (2003) crustal model as the variance.

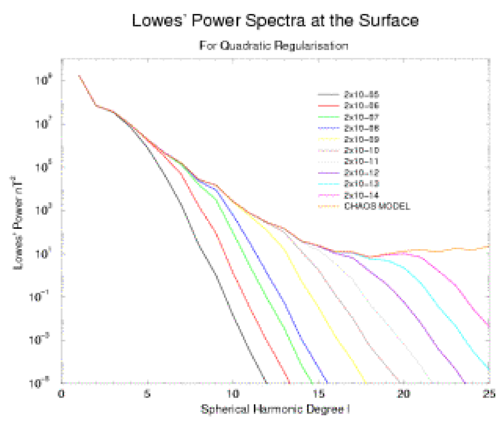


Figure 4 is using the quadratic regularisation method.

Magnetic Field Spectra

Figures 2-4 show the Lowes' power spectrum of different model solutions for different core magnetic field mapping techniques. Figures 5 and 6 show results of spectra for the two Maximum Entropy techniques. The example solutions are models with very similar spectra to the preferred model of Jackson (20003).

- The Maximum Entropy technique with squared misfit produces the smoothest spectra (figs. 2-4)
- The crustal variance technique fits the data more closely at large/intermediate l but the power falls more steeply at higher l. This can be seen in figures 5 and 6 as the predicted crustal spectrum (misfit) is lower at small l using the crustal variance technique than the squared misfit for similar final models.
- The two chosen models which are comparable with Jackson's (2003) model have spectra that lose a lot of power in the large/intermediate scales (l = 6-11). This loss of power is unrealistic as it is believed almost all the power in these harmonics is due to the core field.
- The quadratic regularisation technique tapers the power very rapidly. The starting point of this taper depends on the damping parameter used.

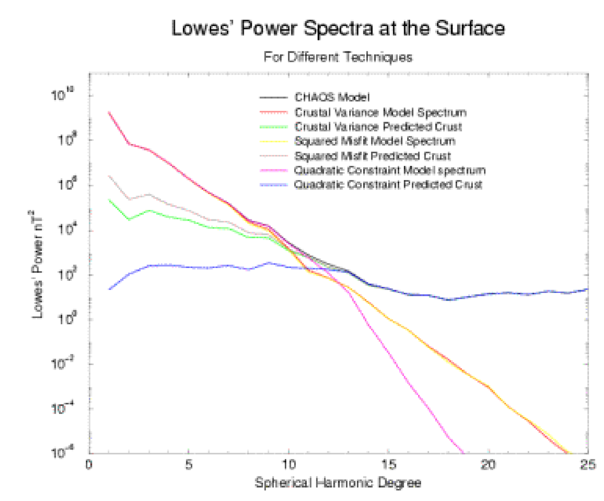


Figure 5 illustrates difference in misfit (crustal prediction) for different Entropy techniques at the surface.

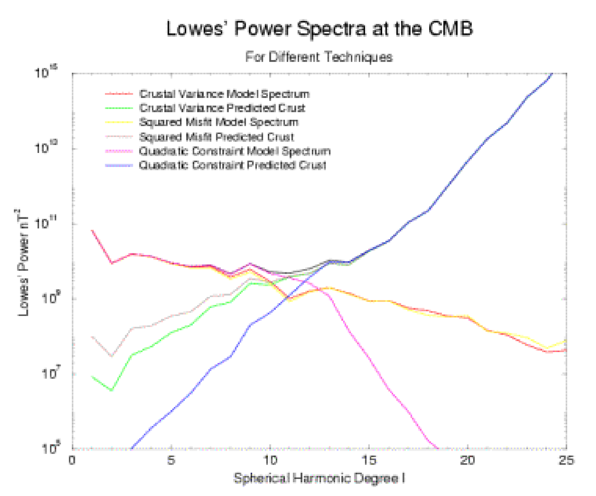


Figure 6 illustrates difference in misfit (crustal prediction) for different Entropy techniques at the CMB.

Core Field Maps

The three maps displayed in figures 7-9 are the quadratic regularisation technique with damping parameter 2×10^{-09} and the two solution models for the spectra of the two Entropy techniques displayed in figures 5 and 6.

- The Entropy models have many more distinct peaks. This can be seen in many areas but most distinctly in the top left of the maps, where one peak in the quadratic regularisation has been resolved into 3 different peaks in the Entropy maps.
- Due to the higher resolution, the Entropy maps have more area where the magnetic field is low but the more numerous peaks also have higher maximum amplitudes. This type of map is more common in dynamo output.
- The differences in the maps of the two Entropy techniques are less apparent

Figure 7. Core Field Map for quadratic regularisation, damping = 2×10^{-09}

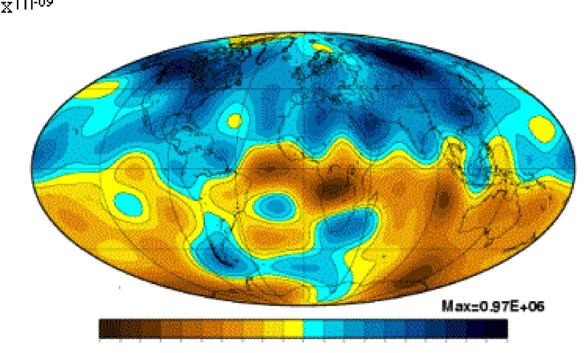


Figure 8. Core field map using Maximum Entropy with squared misfit.

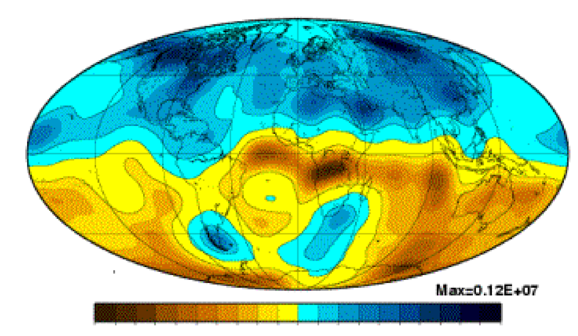
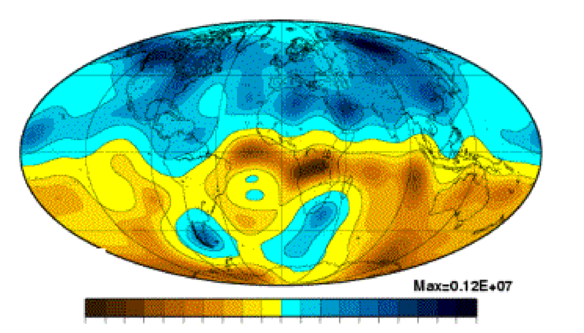


Figure 9. Core field map using Maximum Entropy with crustal variance.



Alternative Choice of Model

As shown above many different final solutions are possible using both the Entropy techniques. This leaves the decision of which model to choose.

Voorhies (2004) discusses the shape of the magnetic field spectra at the CMB. Several predictions were of a form similar to

$$R_l(CMB) = \frac{2K_M}{(2l+1)}$$

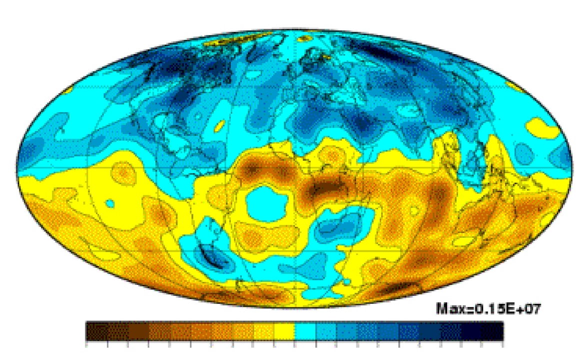
Using this definition the spectra of the model solutions were multiplied by $2l$+1. The spectra which then became white i.e. a constant $2K_M$, were selected and the field maps can be seen in figures 11 and 12.

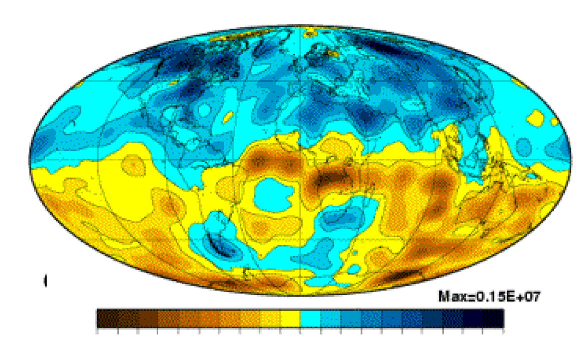
The field maps are similar for both techniques but are different from figures 8 and 9. The features of the Maximum entropy technique are exaggerated. The peaks are even more numerous and have higher maximum values. Even more of the maps are of low level B_r with better resolved maxima, therefore the maps have stronger gradients. The stronger gradients seem to allow more reverse field especially in the southern hemisphere.

Figure 10 (below) shows the spectra of the chosen models that produce white spectra when multiplied by $2l$+1.

Figures 11 (right) is the field map for the crustal variance technique.

Figure 12 (below right) is the field map for the squared misfit.





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Conclusions

- Maximum Entropy imaging can produce fine scale detail in core field models.
  - It can be applied to high quality satellite models which have already removed the external magnetic fields signal.
  - In this form the calculation is simple – involves only Gauss coefficients and not data.
- Many different spectra can be produced by just changing the damping parameter.
  - More realistic spectra can be produced using constraints such as used by Voorhies (2004) to choose the most "Earth Like" model.
  - These spectra have power in the interesting intermediate harmonics where there is thought to be a comparable contribution from crust and core.
- The Field Maps have more influence from the intermediate and small scales thus potentially more information in them.
  - Gradients in magnetic field are stronger using the Maximum Entropy technique.
  - There are more numerous and better resolved peaks.
  - There are larger areas with low intensity magnetic field.
  - Higher overall maxima in magnetic field.

Issues

- Is the field, at higher l, which is suppressed by the Maximum Entropy technique crustal field or is what is lost part of the core field?
- Method strength: Many different solutions can be found
- Method weakness: Which one to choose?!
- The misfit to the data still does not form a spectrum that looks like the predicted crustal field spectrum.

Future Work

- Look at effect of changing the default map, perhaps a better default would be a dipole?
- Look into ways of choosing a final model.

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