Planetary Geodynamics GSFC 698
Studying the Solid Earth and Planets from Space

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High resolution lithospheric field recovery using gradient data within a CM framework as applied to the Swarm mission simulation

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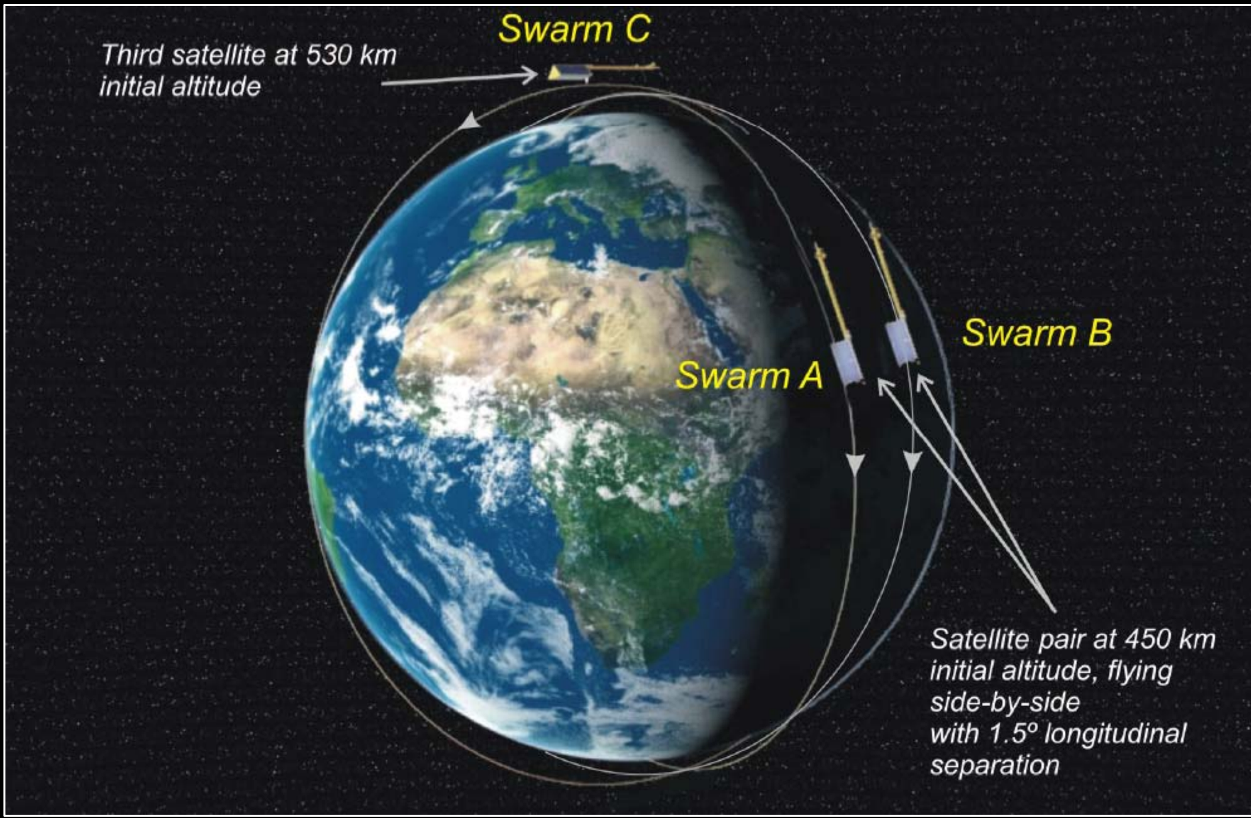
Outline

- Introduction to E2E simulator
- Serially correlated error
- Non-zero mean systematic error
- Exploiting *Swarm* gradient data

Swarm end-to-end simulation data



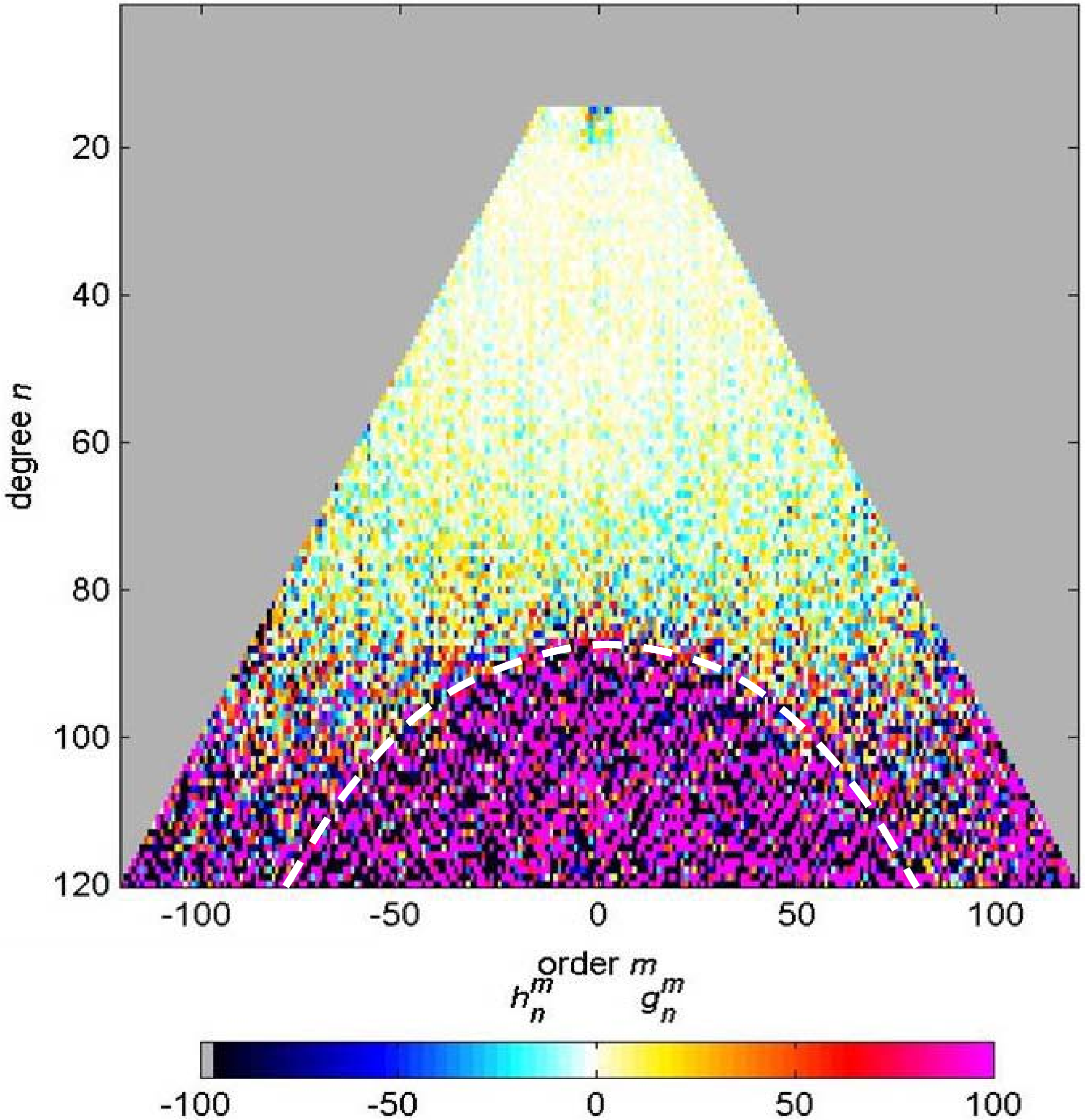
- Synthesized for 1997-2002
- Initially, 1 min sampling interval

- Forward model:
  - Core / SV from CM4
  - Crust from CM4 (low *n*), MF2 (mid *n*), and synthetic (high *n*)
  - Magnetosphere / induced from observatories / 3-D
  - Ionosphere / induced from CM4
  - Realistic instrument noise

Analysis of end-to-end data

- Core / Lithosphere / Sv
  - n_{max} = 13 (core)
  - n_{max} = 120 (crust)
  - B-spline basis for SV
- Magnetosphere
  - n_{max} = 3, m_{max} = 1
  - 1 hr bins
- Ionosphere / Induced
  - QD symmetry
  - $F_{10.7}$ 3-monthly means
  - 1-D radial conductivity
- High-frequency induced
  - n_{max} = 3, m_{max} = 3
  - 1 hr bins
  - point-wise uncorrelated with SV

High-resolution lithospheric expansions and along-track aliasing

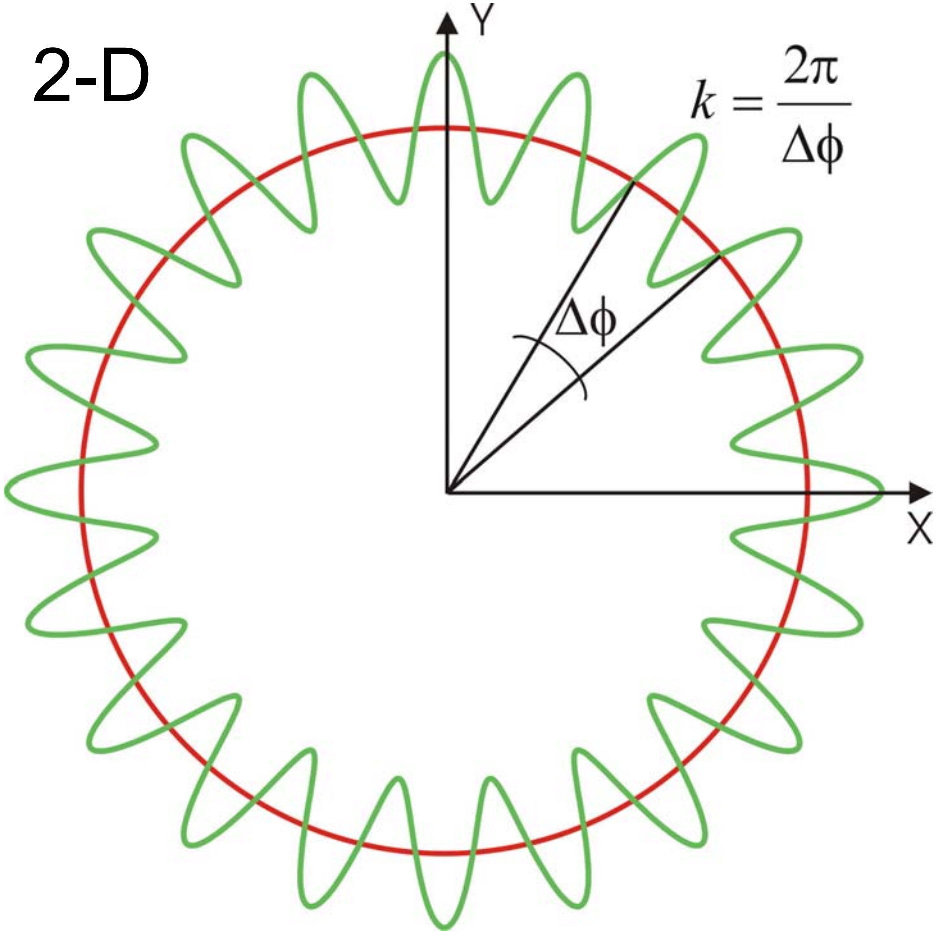


$$S(n,m) = 100 \frac{\tilde{c}_n^m - c_n^m}{\sqrt{\frac{1}{2n+1}\sum_{m=-n}^{n}(c_n^m)^2}}$$

- Hemispherical pattern centered on m=0
- Band-limited to n ≥ 90

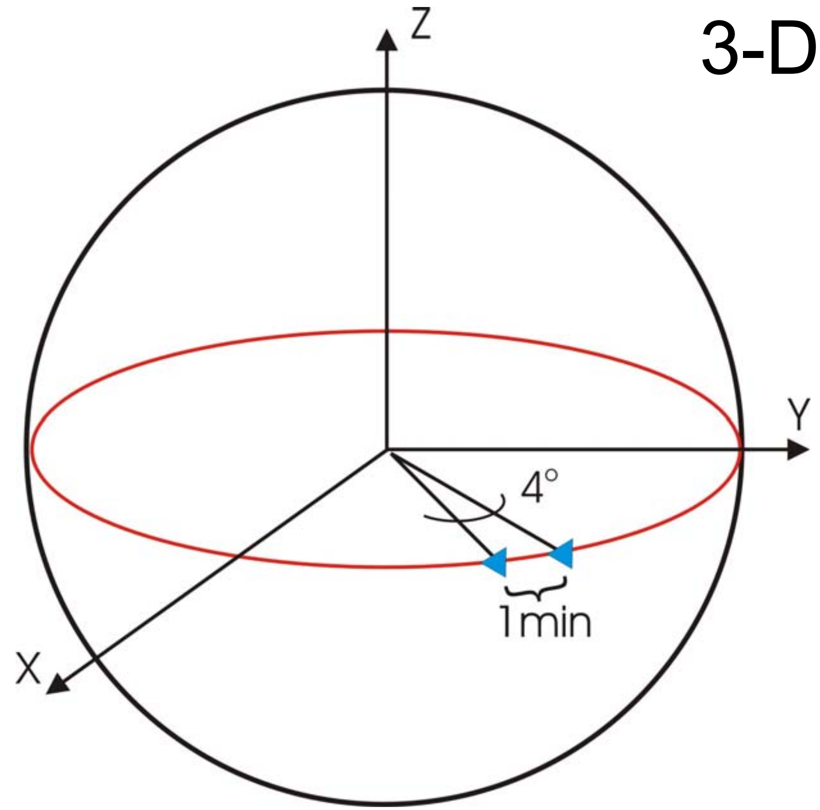
High-resolution lithospheric expansions and along-track aliasing

2-D



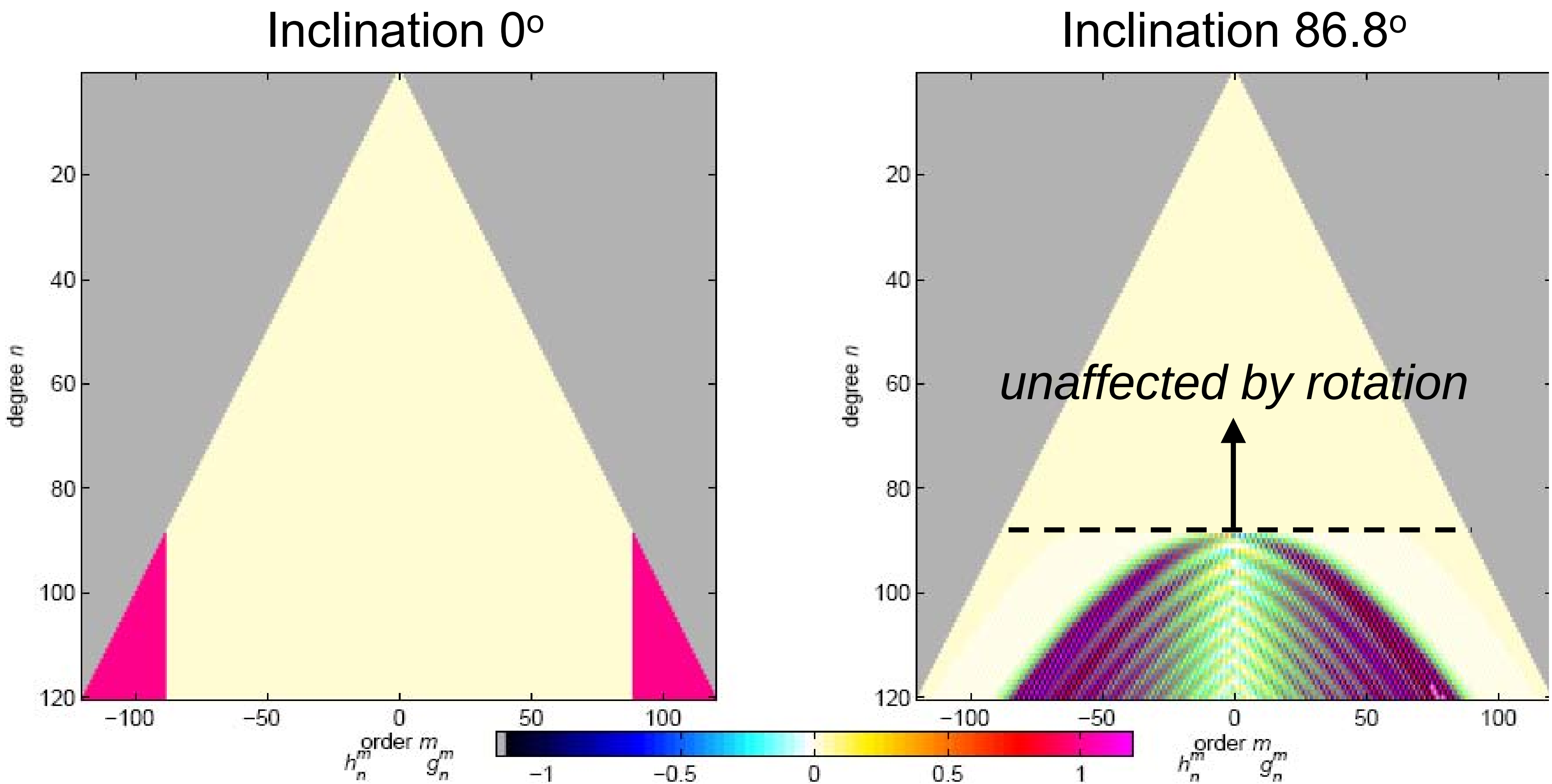
$e^{im\phi}$ unconstrained for $m \geq k$

3-D

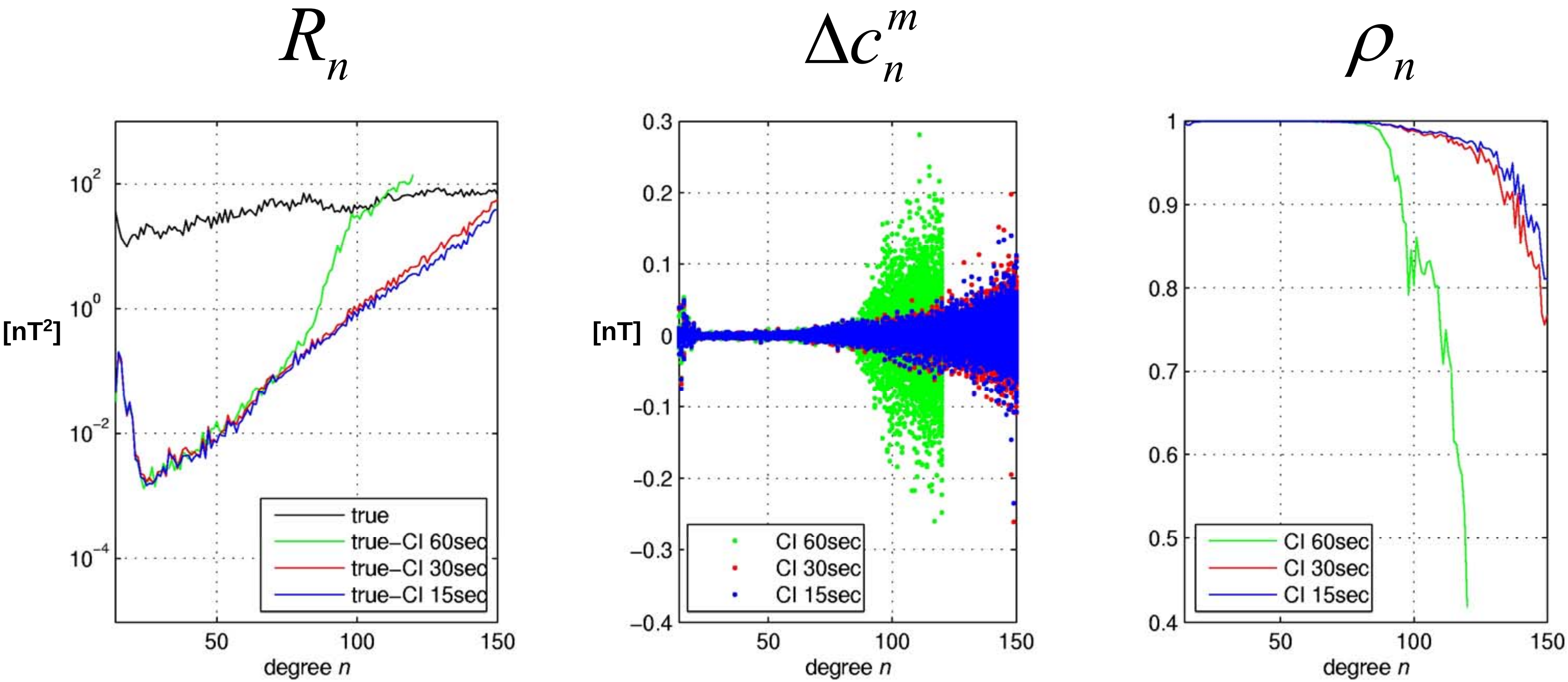


- 90 min *Swarm* orbital period
- $Y_n^m(\theta,\phi)$ unconstrained for $m \geq 90$ in equatorial orbit

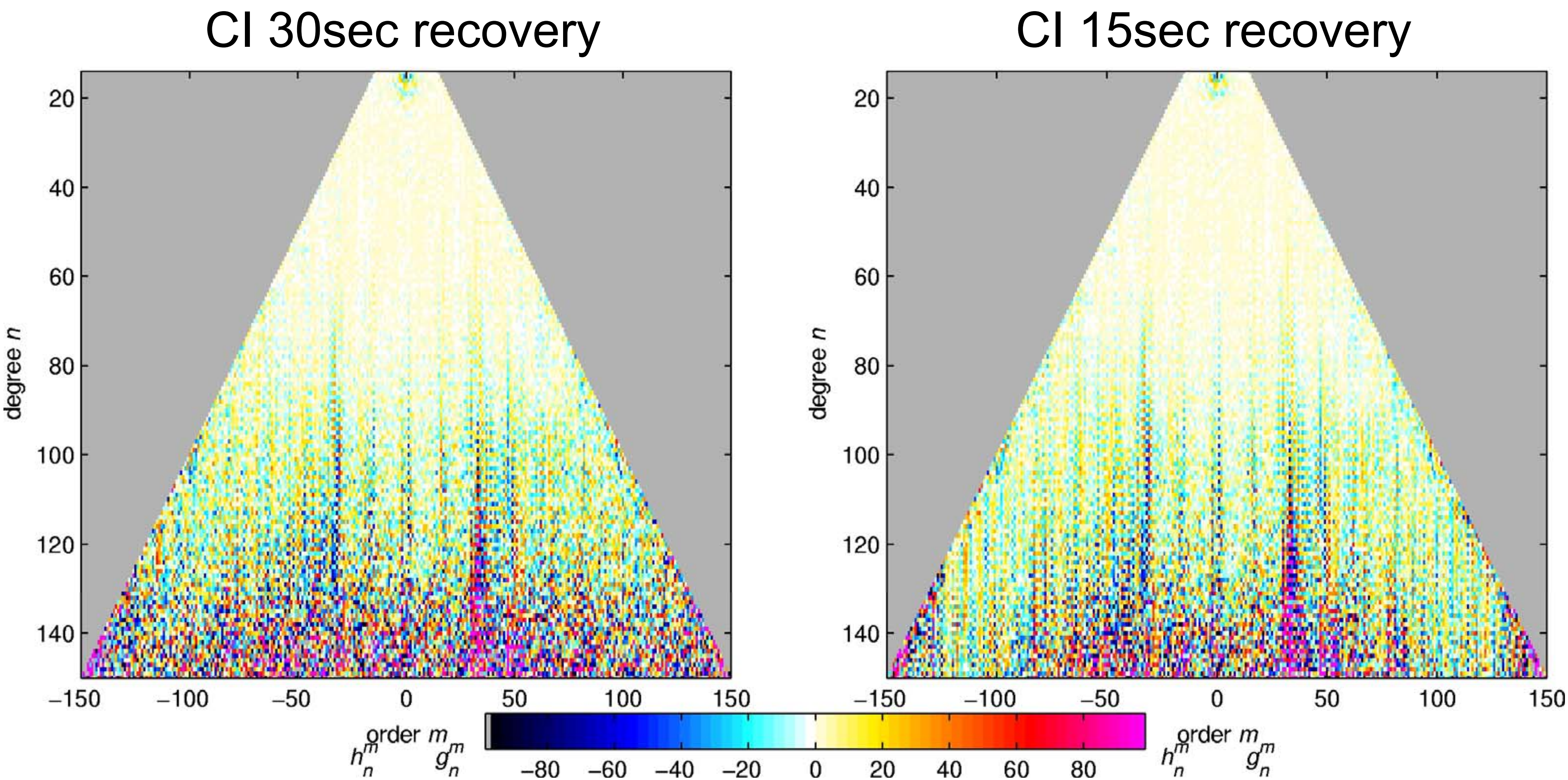
High-resolution lithospheric expansions and along-track aliasing



High-resolution lithospheric expansions and along-track aliasing



High-resolution lithospheric expansions and along-track aliasing



Least squares and systematic noise with non-zero mean

Consider the following linear model

$$\underline{d} = \mathrm{A}\underline{x} + \underline{\nu}$$

where the noise vector

$$\underline{\nu} = \mathrm{B}\underline{z} + \underline{\eta}$$

has uncorrelated systematic and random parts, such that

$$\left.\begin{array}{l}\underline{\eta} = \mathrm{N}(\underline{0}, \mathrm{C}) \\ \underline{z} = \mathrm{N}(\underline{\mu}, \mathrm{Q})\end{array}\right\} \Rightarrow \underline{\nu} = \mathrm{N}(\mathrm{B}\underline{\mu}, \mathrm{C} + \mathrm{BQB}^{\mathrm{T}})$$

Least squares and systematic noise with non-zero mean

Under the least squares assumption of zero-mean noise

$$W = (C + BQB^T)^{-1}$$

which gives the following estimate

$$\begin{aligned}\underline{\tilde{x}} &= (A^T W A)^{-1} A^T W \underline{d}, \\ &= \underline{x} + (A^T W A)^{-1} A^T W (B\underline{z} + \underline{\eta})\end{aligned}$$

which is biased for non-zero μ

$$E[\underline{\tilde{x}}] = \underline{x} + (A^T W A)^{-1} A^T W B \underline{\mu}$$

Least squares and systematic noise with non-zero mean

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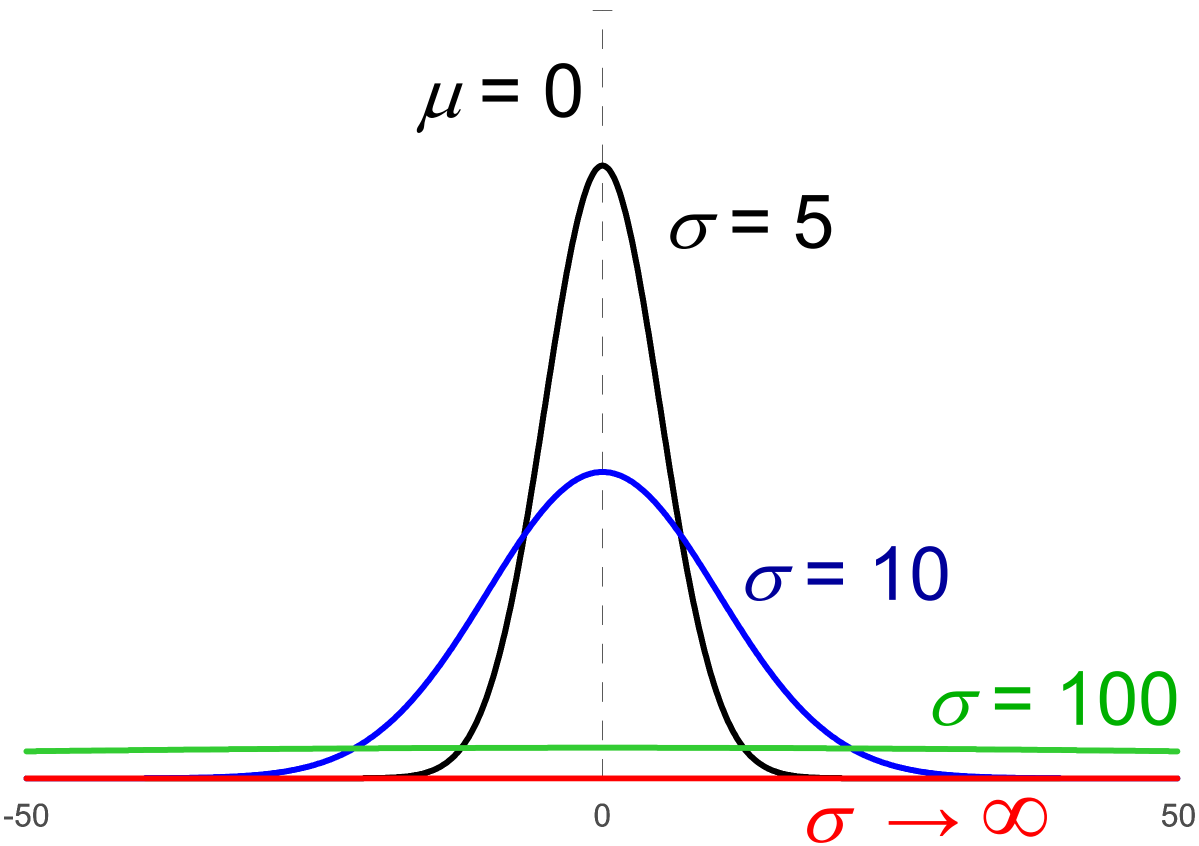
Gaussian noise pdf



Least squares assumes

- $\mu = 0$ for $\sigma < \infty$

But

- $\mu \rightarrow$ *und* for $\sigma \rightarrow \infty$

Infinite-variance weighting (IVW)

This suggests using infinite-variances in weighting non-zero mean systematic noise, and so let

$$Q = \sigma^2 \tilde{Q}$$

and define

$$\begin{aligned} W_\infty &\equiv \lim_{\sigma^2 \to \infty} W, \\ &= \lim_{\sigma^{-2} \to 0} C^{-1} - C^{-1}B(\sigma^{-2}\tilde{Q}^{-1} + B^T C^{-1} B)^{-1} B^T C^{-1}, \\ &= C^{-1} - C^{-1}B(B^T C^{-1} B)^{-1} B^T C^{-1} \end{aligned}$$

IVW

Note that

$$W_\infty B = C^{-1}B - C^{-1}B(B^T C^{-1} B)^{-1} B^T C^{-1} B = 0$$

and so the least squares estimate is now

$$\underline{\tilde{x}} = (A^T W_\infty A)^{-1} A^T W_\infty \underline{d},$$
$$= \underline{x} + (A^T W_\infty A)^{-1} A^T W_\infty \underline{\eta}$$

which is unbiased, regardless of the state of $\underline{z}$

$$E[\underline{\tilde{x}}] = \underline{x}$$

IVW

…but is equivalent to ***co-estimating*** $\underline{z}$ with $\underline{x}$ from the augmented system (*Sabaka and Olsen, EPS, 2006*)!

$$\underline{d} = \begin{pmatrix} \mathrm{A} & \mathrm{B} \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{z} \end{pmatrix} + \underline{\eta}$$

Advantages of IVW with co-estimation

- No knowledge of Q required
- Each realization of B$\underline{z}$ eliminated, not just the average
- Dense, data-by-data W_∞ not explicitly formed

Selective IVW

- Typically in physical systems, the S/N ratio for a given parameter varies with data type
- Selective **IVW** exploits this by recombining data and eliminating systematic noise from less sensitive subsets
- This not only eliminates systematic noise in the mean, but also in each realization, which is really what is present in any given measurement

Selective IVW

Now consider a linear model with two data subsets and two parameter subsets

$$\begin{pmatrix} \underline{d}_1 \\ \underline{d}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{A}_1 & \mathrm{B}_1 \\ \mathrm{A}_2 & \mathrm{B}_2 \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} + \begin{pmatrix} \underline{\eta}_1 \\ \underline{\nu}_2 \end{pmatrix}$$

where

$$\underline{\nu}_2 = \mathrm{B}_2 \underline{z} + \underline{\eta}_2$$

note that $\underline{z}$ represents a systematic contamination of $\underline{y}$ in the $\underline{d}_2$ data subset

Selective IVW

If $\underline{\eta}_1$, $\underline{\eta}_2$, and $\underline{z}$ are uncorrelated, then

$$\underline{\eta}_1 = \mathrm{N}(\underline{0}, \mathrm{C}_1)$$

$$\left.\begin{array}{l} \underline{\eta}_2 = \mathrm{N}(\underline{0}, \mathrm{C}_2) \\ \underline{z} = \mathrm{N}(\underline{\mu}, \mathrm{Q}) \end{array}\right\} \Rightarrow \underline{\nu}_2 = \mathrm{N}(\mathrm{B}_2 \underline{\mu}, \mathrm{C}_2 + \mathrm{B}_2 \mathrm{Q} \mathrm{B}_2^\mathrm{T})$$

Letting $\mathrm{Q} \rightarrow \infty$ leads to the following weight matrix

$$\mathrm{W} = \begin{pmatrix} \mathrm{C}_1^{-1} & 0 \\ 0 & \mathrm{C}_2^{-1} - \mathrm{C}_2^{-1}\mathrm{B}_2(\mathrm{B}_2^\mathrm{T}\mathrm{C}_2^{-1}\mathrm{B}_2)^{-1}\mathrm{B}_2^\mathrm{T}\mathrm{C}_2^{-1} \end{pmatrix}$$

Selective IVW

In terms of co-estimation, the following system is solved

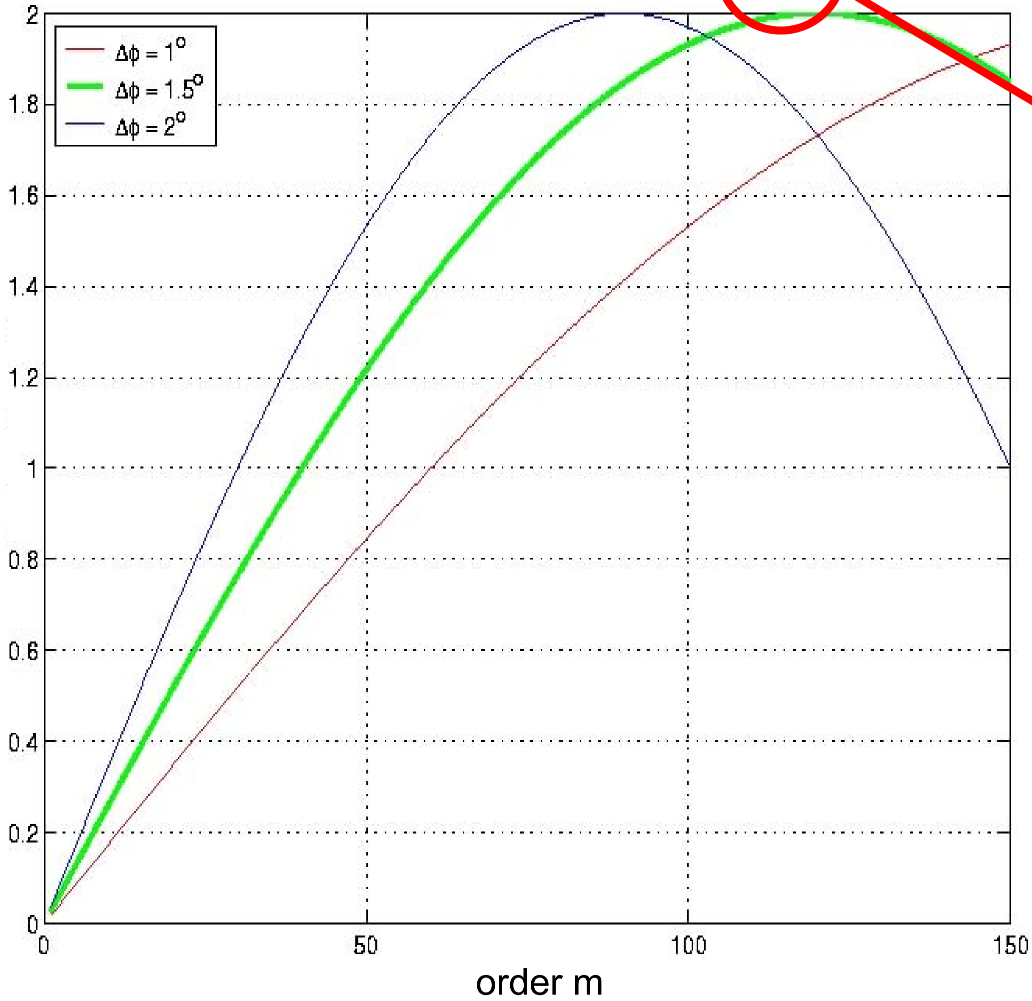
$$\begin{pmatrix} \underline{d}_1 \\ \underline{d}_2 \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & 0 \\ A_2 & B_2 & B_2 \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{pmatrix} + \begin{pmatrix} \underline{\eta}_1 \\ \underline{\eta}_2 \end{pmatrix}$$

with weight matrix

$$W = \begin{pmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{pmatrix}$$

Gradient measurements from *Swarm*

Ideal gain factor $|K_m|$



Internal potentials

$$V_{fld} = \frac{1}{2}(V_{diff} + V_{sum})$$

where

$$V_{fld}(r,\theta,\phi) = a\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=-n}^{n} c_n^m Y_n^m(\theta,\phi)$$

$$V_{diff}(r,\theta,\phi) = a\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=-n}^{n} c_n^m K_m Y_n^m(\theta,\phi)$$

$$V_{sum}(r,\theta,\phi) = a\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=-n}^{n} c_n^m L_m Y_n^m(\theta,\phi)$$

such that

$$|L_m| = 2 - |K_m|$$

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Application to *Swarm* data in the CM framework

- Let $\underline{d}_1$, $\underline{d}_2$, and $\underline{d}_3$ be the vector data from *Swarm* low satellites 1 and 2 and high satellite 3
- Assume $\dim(\underline{d}_1) = \dim(\underline{d}_2)$, and that their elements are chronologically matched
- Let $\underline{x}$ be non-crustal, and $\underline{y}_\ell$ and $\underline{y}_h$ be low-order ($m \leq 20$) and high-order ($m > 20$) crustal field parameters
- Let $\underline{\nu}_1$, $\underline{\nu}_2$, and $\underline{\nu}_3$ be the noise vectors associated with $\underline{d}_1$, $\underline{d}_2$, and $\underline{d}_3$, respectively

Application to *Swarm* data in the CM framework

Rotating the pre-whitened system equations

$$\mathsf{R}\begin{pmatrix} \underline{d}_1 \\ \underline{d}_2 \\ \underline{d}_3 \end{pmatrix} = \mathsf{R}\left[\begin{pmatrix} \mathsf{A}_1 & \mathsf{B}_1^{\ell} & \mathsf{B}_1^{h} \\ \mathsf{A}_2 & \mathsf{B}_2^{\ell} & \mathsf{B}_2^{h} \\ \mathsf{A}_3 & \mathsf{B}_3^{\ell} & \mathsf{B}_3^{h} \end{pmatrix}\begin{pmatrix} \underline{x} \\ \underline{y}_{\ell} \\ \underline{y}_{h} \end{pmatrix} + \begin{pmatrix} \underline{\nu}_1 \\ \underline{\nu}_2 \\ \underline{\nu}_3 \end{pmatrix}\right]$$

where

$$\mathsf{R} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

Application to *Swarm* data in the CM framework

gives

$$\begin{pmatrix} \underline{d}_{\mathrm{s}} \\ \underline{d}_{\mathrm{d}} \\ \underline{d}_{3} \end{pmatrix} = \begin{pmatrix} \mathrm{A}_{\mathrm{s}} & \mathrm{B}^{\ell}_{\mathrm{s}} & \mathrm{B}^{h}_{\mathrm{s}} \\ \mathrm{A}_{\mathrm{d}} & \mathrm{B}^{\ell}_{\mathrm{d}} & \mathrm{B}^{h}_{\mathrm{d}} \\ \mathrm{A}_{3} & \mathrm{B}^{\ell}_{3} & \mathrm{B}^{h}_{3} \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{y}_{\ell} \\ \underline{y}_{h} \end{pmatrix} + \begin{pmatrix} \underline{\nu}_{\mathrm{s}} \\ \underline{\nu}_{\mathrm{d}} \\ \underline{\nu}_{3} \end{pmatrix}$$

where the subscripted “s” and “d” indicate sums and differences, respectively

Application to *Swarm* data in the CM framework

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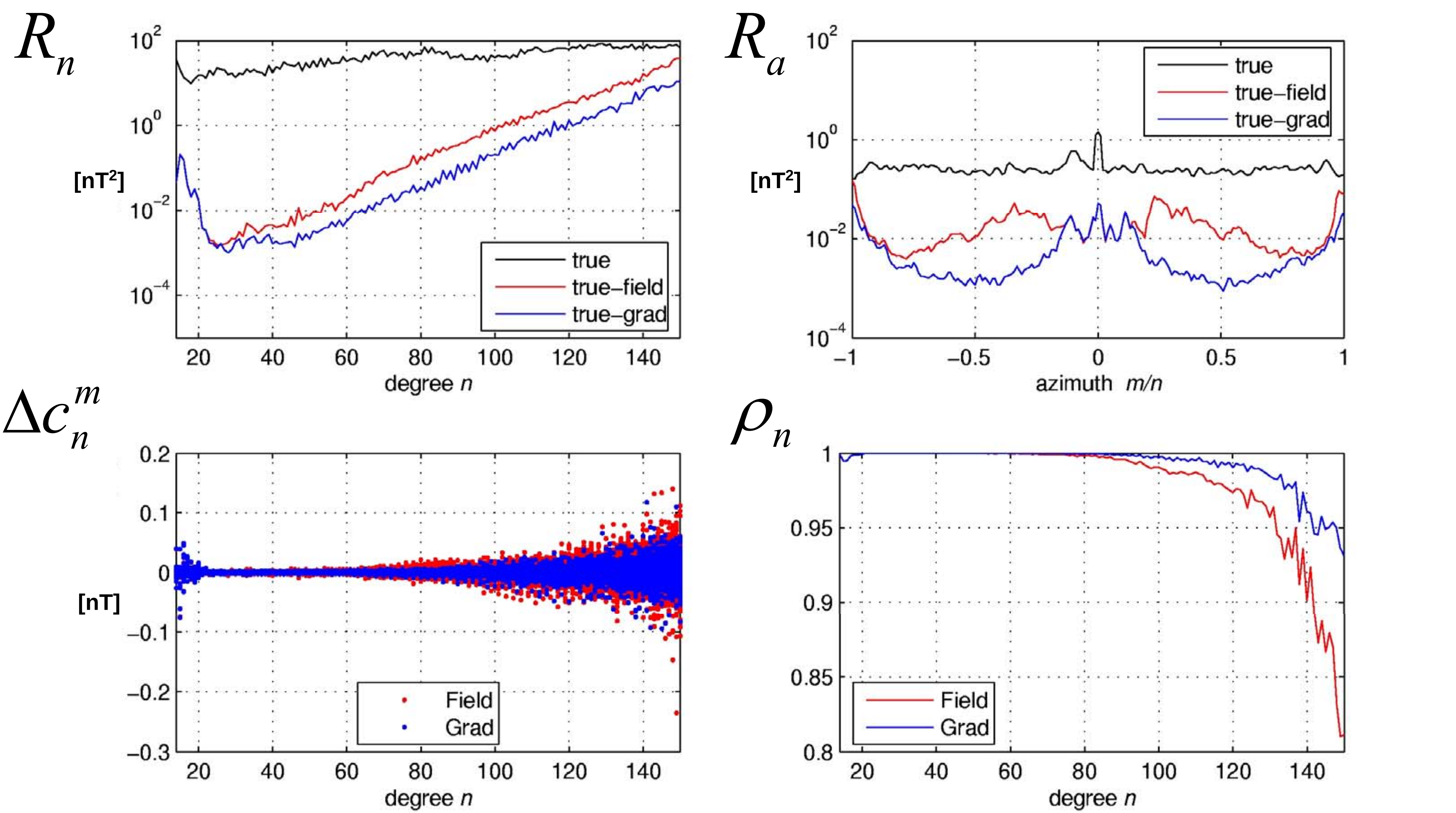
Assuming $\underline{d}_s$ and $\underline{d}_3$ are more sensitive to $\underline{y}_\ell$, and $\underline{d}_d$ is more sensitive to $\underline{y}_h$ leads to the following working CM system

$$\begin{pmatrix} \underline{d}_s \\ \underline{d}_d \\ \underline{d}_3 \end{pmatrix} = \begin{pmatrix} A_s & B_s^\ell & B_s^h & 0 & B_s^h \\ A_d & B_d^\ell & B_d^h & B_d^\ell & 0 \\ A_3 & B_3^\ell & B_3^h & 0 & B_3^h \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{y}_\ell \\ \underline{y}_h \\ \underline{z}_\ell \\ \underline{z}_h \end{pmatrix} + \begin{pmatrix} \underline{\eta}_s \\ \underline{\eta}_d \\ \underline{\eta}_3 \end{pmatrix}$$

likely due to unmodelled, time-varying external / induced fields

random

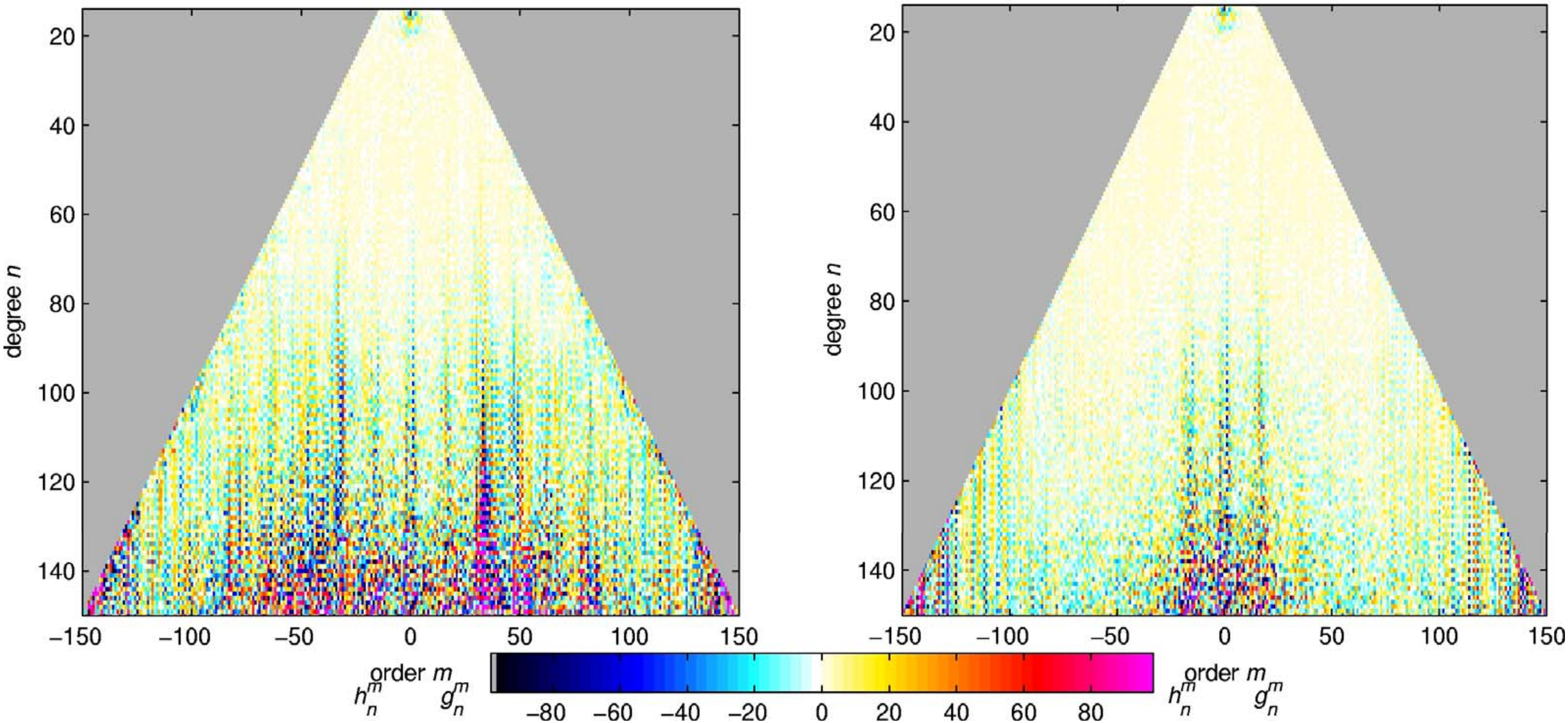
Application to *Swarm* data in the CM framework



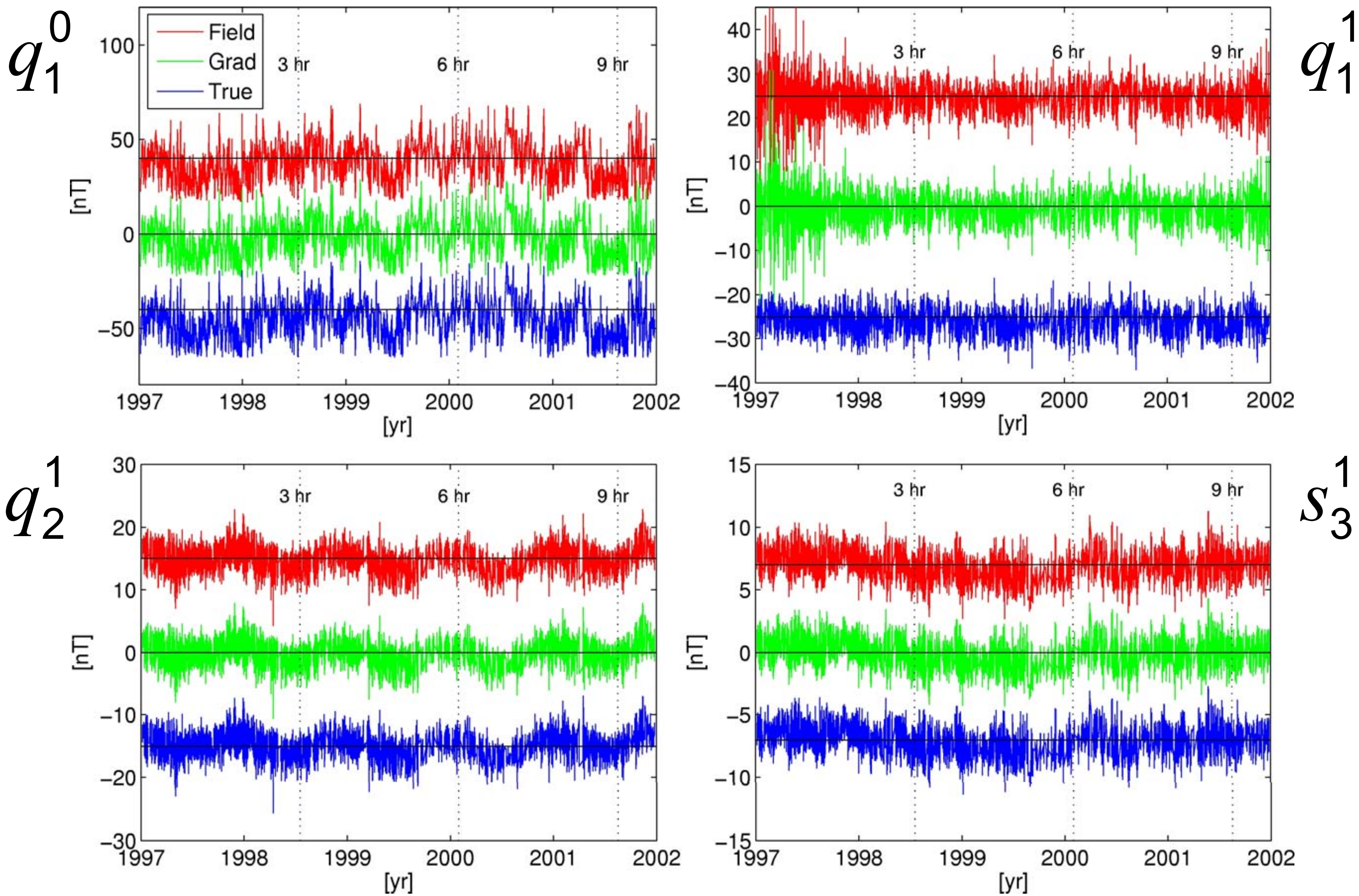
Application to *Swarm* data in the CM framework

Field recovery

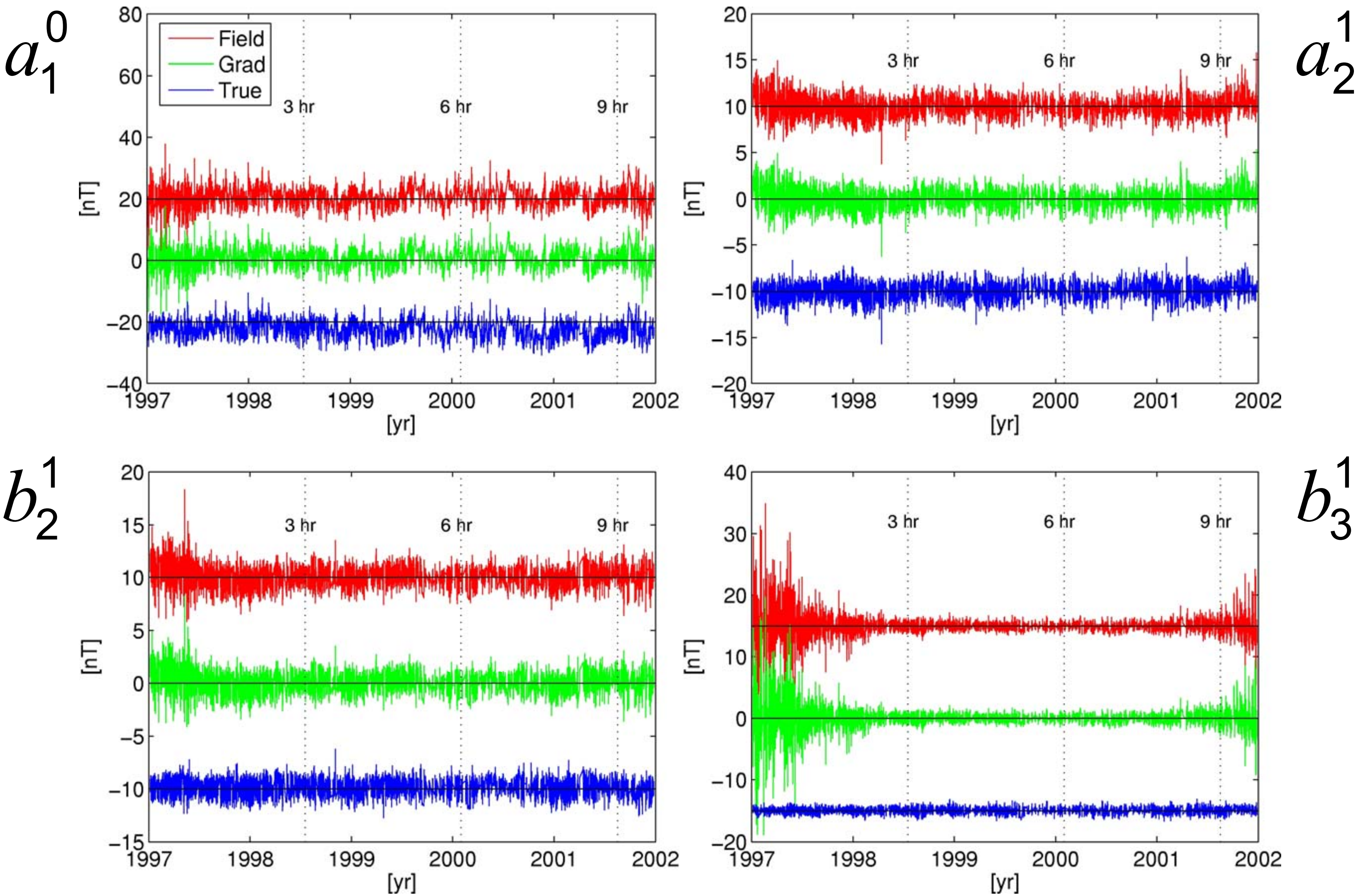
Gradient recovery



Application to *Swarm* data in the CM framework



Application to *Swarm* data in the CM framework



Conclusions

- For high-degree lithospheric modelling, higher sampling rates may be needed to reduce along-track aliasing

- Unmodelled natural fields are a common source of non-zero mean systematic noise and **IVW** is able to mitigate its effect while typical weighting does not

- Selective **IVW** combines the strengths of the CM approach with those of data selection and filtering

Conclusions

- Treating the rotated *Swarm* data with selective **IVW** produces superior lithospheric field models compared to treating straight data with typical least squares

- Non-lithospheric field models remain essentially unchanged